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Truncation and reset process on the dynamics of Parrondo's games

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The counter-intuitive feature of Parrondo's games is illustrated on various dynamical systems combined from different deterministic and stochastic subsystems. The concept of truncation and reset process is introduced, which provides a transparent perspective to understand the underlying mechanism of this class of dynamics, including the transport of flashing ratchets, and clarifies the puzzlement why random switching between two games can generate reversal dynamics as periodical switching does.

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Parrondo's games are considered counter intuitive [1–3]. As stated by Parrondo *et al.* in Ref. [1] "the apparent paradox points out that if one combines two dynamics in which a given variable decreases, the same variable can increase in the resulting dynamics." This statement has caught immense public interest and brought about broad ranges of discussion in different disciplines, including mating pattern in evolution [3], investment strategy [4], biased Brownian motion by flashing ratchet mechanism [3], pattern formation [5], quantum games [6], and games in Ising and Potts models [7]. The most dominant feature in Parrondo's games is the reversal dynamics after two subsystems are combined. However, this feature itself is not unexpected, because a combined system usually exhibits different behavior if the subsystems are coupled, as the examples in Fig. 2 show. Instead of that, an essential question on these games is at which point these two subsystems are coupled and how this unapparent and weak couple can change the dynamics dramatically. This question is the main concern of this work and is investigated on various dynamical systems combined from deterministically chaotic and stochastic subsystems. Therein, the concept of truncation and reset process is introduced, which provides a transparent perspective to understand the underlying mechanism of the combined dynamics and clarifies the frequent puzzlement why random switching between two subsystems can induce new biased dynamics, as well as periodical switching.

The coin-tossing model (CTM) frequently mentioned in Parrondo's games consists of three biased coins with the probabilities of winning $P_{\rm win}$ and losing $P_{\rm loss}$

	Coin 1	Coin 2	Coin 3
$P_{\rm win}$	$1/2-\varepsilon$	$3/4-\varepsilon$	$1/10-\varepsilon$
$P_{\rm loss}$	$1/2+\varepsilon$	$1/4+\varepsilon$	$9/10+\varepsilon$

where ε is a small number and winning (losing) means that the player receives (loses) one dollar in his capital $X(n)$ in the nth tossing. The coin used in game A and B for the nth tossing is determined by the game rules:

Games	Coins	Condition to use this coin
A	coin 1	
B	coin 2	if $X(n-1)$ mod $3=0$
	coin 3	if $X(n-1)$ mod $3=1,2$.

That is, in game A only coin 1 is used. In game B either coin 2 or coin 3 is used depending on, respectively, whether the previous capital $X(n-1)$ is a multiple of 3 or not. Analytical estimation and simulation results [2] show that both games are losing games for a small $\varepsilon>0$, when they are played individually. However, the player wins when switching between two games.

At first sight, this phenomenon is counter intuitive because the coin-tossing processes in game A and B are both stochastic and should be independent. Intuitively, the chance of winning in the combined dynamics should equal to the average of the two individuals. However, a closer look at the rules shows that these two individual game dynamics in the combined game are not completely independent. In contrast to game A, which is independent on game B after the combination, game B is not independent on game A, because of the condition of multiple 3. This condition provides a weak couple between these two games and affects the whole dynamics. To illustrate this coupled effect, the deterministic condition of multiple 3 in game B of the CTM is replaced with a stronger condition, given by the deterministic maps (2) and (5).

The first dynamics considered is the map $T_A:[0,1]\to[0,1]$, with

$$T_A x=\begin{cases}\varepsilon_A & \text{for } x\in[0,\frac{1}{2}],\\ 1-\varepsilon_A & \text{for } x\in[\frac{1}{2},1],\end{cases} \tag{1}$$

where $0<\varepsilon_A<1$ [see example for $\varepsilon_A=0.9$ in Fig. 1(a)]. We call it step map because this dynamics has only two values ε_A and $1-\varepsilon_A$, depending on in which region the point x is located. The second dynamics is the circle map $T_B:[0,1]\to[0,1]$, with

$$T_B x=x+\varepsilon_B \mod 1, \tag{2}$$

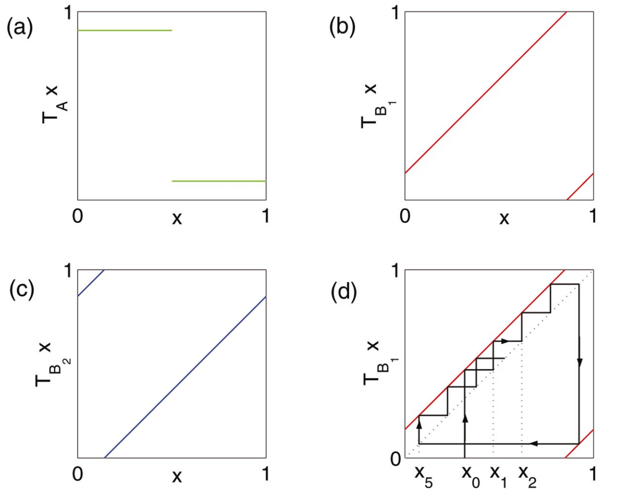

FIG. 1. (Color) (a) Step map T_A with $\varepsilon_A = 0.9$. (b) Circle map T_{B_1} with $\varepsilon_{B_1} = \pi - 3$. (c) Circle map T_{B_2} with $\varepsilon_{B_2} = 4 - \pi$. (d) A route of T_{B_1} with the initial point x_0.

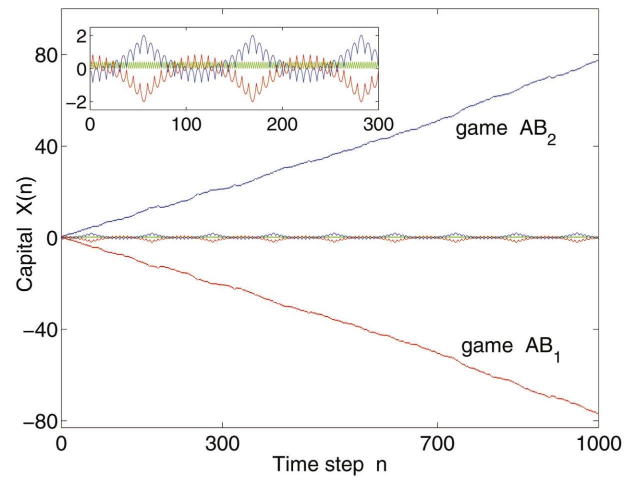

FIG. 2. (Color) The capital $X(n)$ of game AB_1 (red curve with negative slope) and game AB_2 (blue curve with positive slope). The capital $X(n)$ of individual game A (green), game B_1 (red), and game B_2 (blue) are magnified in the inset.

where $0 < \varepsilon_B < 1$ is an irrational number [see examples for $\varepsilon_B = \pi - 3$ and $4 - \pi$ in Fig. 1(b,c)]. Given an initial point $x_0 \in [0,1]$, these maps T generate an orbit $\{x_0, x_1, x_2, \ldots\}$ with $x_1 = Tx_0$, $x_2 = T^2 x_0, \ldots$. The number x_n corresponds to the coin value in the nth tossing. Notably, the value x_n in coin tossing is stochastic and not related to x_{n-1}, while the value x_n here depends on x_{n-1} and is therefore deterministic. The game rule is such that one gains $s_n = x_n - \frac{1}{2}$ dollar for an outcome x_n. Since $x_n \in [0,1]$, the gain $s_n \in [-\frac{1}{2}, \frac{1}{2}]$ can be both positive and negative. The new capital $X(n)$ changes from the previous amount $X(n-1)$ by $X(n) = X(n-1) + s_n$. Game A and B are generated by dynamics T_A and T_B, respectively, and game AB is generated by dynamics switching between T_A and T_B randomly.

The individual game A is a fair game because x_n, $n = 0, 1, 2, \ldots$, oscillates between ε_A and $1 - \varepsilon_A$ with an average of $\frac{1}{2}$. The capital $X(n)$ does not increase for large n, as plotted by the green curve in the inset of Fig. 2. The individual game B is also a fair game, because T_B is unique ergodic with Lebesque measure as the ergodic measure [8]. That is, the points in an orbit spread uniformly in the interval $[0,1]$. Therefore, the average value of x_n and s_n are $\frac{1}{2}$ and 0, respectively, for infinite time iteration. The oscillating capital $X(n)$ is bounded close to zero and does not increase for large n, as plotted by the red and blue curves for T_{B_1} and T_{B_2} in the inset of Fig. 2. However, the combined game AB is no longer a fair game. For the ε_A, ε_{B_1}, and ε_{B_2} values in Fig. 1, the game AB_1 (AB_2) is a losing (winning) game, as plotted by the red (blue) curve in Fig. 2 with negative (positive) slope. Modifying the definition of the gain slightly by changing $s_n = x_n - \frac{1}{2}$ to $s_n = x_n - \frac{1}{2} - \epsilon$ with a small $\epsilon > 0$, it produces the same observation in the CTM that combining two losing games induce a winning game. This simple example reveals that a combined dynamics can behave completely differently from the original individual two.

The dynamics of games A and B in above example are both deterministic, generated by the maps T_A and T_B. However, game A in the CTM is stochastic and game B is pseudostochastic (due to the condition of multiple 3). To study examples closer to the CTM, the deterministic T_A in our game A is replaced by the unbiased stochastic coin-tossing upon two values, $x_n \in \{\varepsilon_A, 1 - \varepsilon_A\}$, which is independent of x_{n-1}. Obviously, the new game A is a fair game and, furthermore, independent on game B. Numerical simulation shows that the increasing and decreasing features of the capital in Fig. 2 do not change after this modification.

The rapid increase or decrease in above capital is easy to understand by tracing the orbit $\{x_0, x_1, x_2, \ldots\}$ of T_B. Figure 1(d) shows an example in T_{B_1}, which shifts x_n step by step from small numbers to large numbers. After x arrives at a largest value, it jumps back to a small number, $x_5 \in [0, \varepsilon_{B_1})$, and forms a cycle of N steps (The value $N \approx 7$ in Fig. 2 depends on ε_{B_1} and x_0). Thereafter, the dynamics repeats similar cycles. Given an initial point x_0 close to zero, the average $(1/m)\Sigma_{n=0}^{m} x_n$ is $\frac{1}{2}$ for $m = \infty$ and around $\frac{1}{2}$ when $m < \infty$ is close to the multiple of a complete cycle, however, smaller than $\frac{1}{2}$ for most other m. This cyclic repetition induces the capital oscillation of game B_1 in Fig. 2. Now, when game B_1 is mixed with game A, the dynamics of T_{B_1} is truncated by T_A continuously, say,

$$\{\underbrace{x_1^A, x_2^{B_1}}_{\text{(i)}}, \underbrace{x_3^A}_{\text{(ii)}}, \underbrace{x_4^A, x_5^{B_1}, x_6^{B_1}, x_7^{B_1}}_{\text{(iii)}}, \underbrace{x_8^A, x_9^{B_1}}_{\text{(iv)}}, \ldots\}, \tag{3}$$

where $x_n^A \in \{\varepsilon_A, 1 - \varepsilon_A\}$ are generated by game A and $x_n^{B_1}$ by game B_1. This sequence is divided into infinite number of segments (i), (ii), (iii), (iv), ... The value x_n^A at the beginning of every segment can be regarded as the initial point of a new orbit of T_{B_1}, reset by game A. Assume first game A is determined by T_A. The segments with the initial point $x_n^A = 1 - \varepsilon_A = 0.1$ have averages less than $\frac{1}{2}$, in general, due to the incomplete cycles mentioned above. The segments with the initial point $x_n^A = \varepsilon_A = 0.9$ will be balanced by the subse-

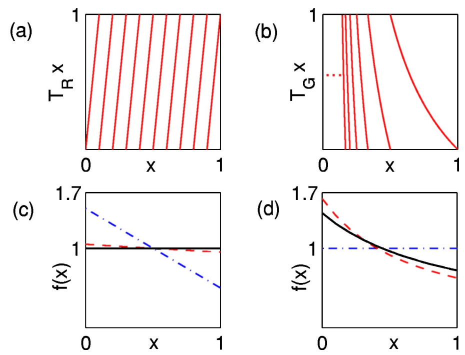

FIG. 3. (a) r-adic map with $r=10$. (b) Gauss map (the dots denote further infinite number of branches). (c) $f(x)=x-\frac{3}{2}$ (dash-dotted line), $\mathcal{L}_R f(x)=\frac{21}{20}-\frac{1}{10}x$ (dashed line), and $f_R^*(x)=1$ (solid line). (d) $f(x)=1$ (dash-dotted line), $\mathcal{L}_R f(x)=\Sigma_{n=1}^{\infty}(1/x+n)^2$ (dashed curve), and $f_G^*(x)=[(x+1)\ln 2]^{-1}$ (solid curve).

quent small value either $x_{n+1}^A=1-\varepsilon_A=0.1$ or $x_{n+1}^B=T_{B_1}\varepsilon_A\approx 0.042$. The rest sequence after these points begins with small values and are less than $\frac{1}{2}$ on average, as in the previous case. Thus, the whole sequence (3) has an average $\langle x^{AB_1}\rangle<\frac{1}{2}$. By analogy, this inequality holds also when game A is the unbiased coin-tossing upon $\{\varepsilon_A, 1-\varepsilon_A\}$. Therefore, the gain average $\langle s\rangle=\langle x^{AB_1}\rangle-\frac{1}{2}$ is negative, which induces the decreasing capital $X(n)$ in Fig. 2. For the increasing $X(n)$ of game AB_2, the argument is similar. The quantity $\langle x^{AB_1}\rangle$ is the average of infinite number of orbit segments of game B_1 truncated by game A, with different initial points of different lengths. This average sensitively depends on the dynamical properties of both games, such as the distribution of the reset points of game A and the orbits of game B_1 after these points. It is not generic that $\langle x^{AB_1}\rangle$ also exactly equals $\frac{1}{2}$, when the uncoupled games A and B_1 have averages $\langle x^A\rangle=\langle x^{B_1}\rangle=\frac{1}{2}$, because they are different mathematical objects. From this perspective, a combined system exhibiting new biased dynamics is generic.

The reset process of game A destroys the dynamical average of the original game B and induces a new biased dynamics. This effect is transparent in the deterministic dynamics T_B because the orbit after a reset point x_n^A is completely determined by this point. However, it is less transparent in the CTM because the orbit after x_n^A is rather stochastic and less related to how x_n^A is reset. Therefore, the reset process analysis on the probability density of the dynamics is necessary for understanding the CTM, which is demonstrated in the following example. The first dynamics considered is the r-adic map $T_R:[0,1]\to[0,1]$, with

$$T_R x=r\,x \bmod 1, \tag{4}$$

where r is a real number [see an example for $r=10$ in Fig. 3(a)]. The second map is the Gauss map $T_G:[0,1]\to[0,1]$, with

$$T_G x=\begin{cases}\dfrac{1}{x} \bmod 1, & x\neq 0\\ 0, & x=0,\end{cases} \tag{5}$$

as shown in Fig. 3(b). These two maps are highly chaotic. The dynamics of a single orbit is unpredictable after few steps of iteration. However, the evolution of an ensemble of initial points, depicted by a probability density, can be described by the Perron-Frobenius (PF) operator $\mathcal{L}_T f(x)=\Sigma_{y\in T^{-1}x}(f(y)/|T'(y)|)$, where the sum runs over the preimage $T^{-1}x=\{y\in I|Ty=x\}$ of T and T' denotes the derivative of T [8,9]. While a map T determines the evolution of a point, its PF operator $\mathcal{L}_T$ determines the evolution of the probability density. The leading eigenvalue of this operator is one. The corresponding eigenfunction $f^*(x)$ is the invariant probability density of the dynamical system, which is the probability of finding the point at position x after infinite number of iterations. Notably, the transfer operator can be regarded as a continuous version of the transition matrix acting on the Markovian chain with three probabilities, corresponding to the three capital values $[X(n) \bmod 3=0,1,2]$, which was used to determine the stationary probability of the CTM [2].

The PF operator for the r-adic map T_R is

$$\mathcal{L}_{T_R} f(x)=\frac{1}{r}\sum_{n=0}^{r-1} f\left(\frac{n}{r}+\frac{x}{r}\right), \tag{6}$$

with the invariant density $f_R^*(x)=1$. That is, any place in $[0,1]$ will be uniformly visited by an initial point after long time iteration of T_R. Therefore, the position average of T_R is $m_R=\frac{1}{2}$. By analogy, the PF operator for T_G is

$$\mathcal{L}_{T_G} f(x)=\sum_{n=1}^{\infty}\left(\frac{1}{x+n}\right)^2 f\left(\frac{1}{x+n}\right). \tag{7}$$

The corresponding invariant density is the Gauss measure $f_G^*(x)=[(x+1)\ln 2]^{-1}$ and the position average is $m_G=(1-\ln 2)/\ln 2\approx 0.44$. Consequently, game A (B) characterized by the gain rule $s_n=x_n-m$, with position average $m=m_R$ (m_G) for T_R (T_G), is a fair game. The capital $X(n)$ in these individual games is bounded close to zero and does not increase or decrease for large n.

The density sequences $\mathcal{L}_R^n f(x)$ and $\mathcal{L}_G^n f(x)$, for $n=1,2,\ldots$, converge to $f_R^*(x)$ and $f_G^*(x)$ rather fast [8], as shown in Fig. 3(c,d). The sequence of the position averages in game AB, say,

$$\{m_1^A,m_2^B,m_3^A,m_4^A,m_5^B,m_6^B,m_7^B,m_8^A,m_9^B,\ldots\}, \tag{8}$$

consists of an ensemble of segments of position average of game B reset by game A. A similar argument as that for sequence (3) leads to the inequality $\langle m^{AB}\rangle>m_G$ and the increasing $X(n)$. This result also holds true for the case when T_R in game A is replaced by the unbiased tossing upon values in $[0,1]$, because then any density is reset to $f^*(x)=1$ immediately after every tossing, which is even faster than the convergence of $\mathcal{L}_R^n f(x)$. Accordingly, the reset process concept can be applied to dynamical systems described by probabilities, including the dynamics in the CTM.

Game A in the CTM is stochastic and its invariant density $f_A^*(x)$ is unity. Game B is pseudostochastic and its invariant density $f_B^*(x)$, in general, is not a constant [2]. When game A and B are mixed, the density tends to converge to $f_B^*(x)$, however, is continuously truncated and reset to $f_A^*(x)$ by game A. The inequality between $\langle m^{AB}\rangle$ and m_B leads to the result that a combination between losing games can be both constructive and destructive, depending on the parameter ε. A constructive combination is then referred to a Parrondo's game. A further feature attracting much attention in Parrondo's games is that an increasing capital $X(n)$ can be generated not only by periodical switching but also by random switching. However, the concept of reset process indicates that an increasing or decreasing $X(n)$ is not necessarily related to periodical or random switching (corresponding to reseting). Consider the simple system in sequence (8) with fast convergent densities $\mathcal{L}_{T_R}^n f(x)$ and $\mathcal{L}_{T_G}^n f(x)$. The average, $\langle m^{AB}\rangle \approx (n_A m_R + n_B m_G)/(n_A + n_B)$, of this dynamics mainly depends on the total numbers n_A and n_B of game A and B but not on how game A and B are switched.

Moreover, the truncation and reset mechanism is apparent in the dynamics of the flashing ratchets [3,10], which is often compared with Parrondo's games. The probability density of the Brownian particle converges to the unity invariant density, when the potential is turned off, because this particle tends to diffuse. However, this density is continuously reset to Dirac's δ function when the potential is turned on. During this procedure, the position average of the Brownian particle is shifted to a certain direction, which induces a biased transport. Similar biased motion in diverse models for motor proteins, such as that in Ref. [11], can also be understood by using the interpretation of this mechanism.

In summary, Parrondo's games belong to a class of dynamical systems in which the dynamics is a combination of two subsystems with an unapparent couple in between. Due to this couple, the combined dynamics can behave completely differently from the uncombined ones. However, because of its unapparentness, this couple is easily overlooked, which leads to the wrong expectation that the combined system should behave like the average of the two uncoupled subsystems. This puzzlement is clarified on various examples with different levels of ambiguity in this work by using the concept of truncation and reset process. Therein, the dynamics of a combined system is regarded as an ensemble of orbits reset to different initial values and truncated to different lengths. Since an averaged quantity of this ensemble and the same averaged quantity of an infinitely long single orbit are different objects and not identical, in general, a combined dynamics exhibiting new behavior is genetic. This concept and examples studied provide a more transparent perspective to understand Parrondo's paradox, including the frequent question why a capital increase can be generated by random switching.

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